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Magneto-Fluid Dynamic Turbulence with a Uniform Imposed Magnetic Field

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Two point correlation and spectral equations for this case are derived from the equations of fluid and electrodynamics. Solutions are obtained by assuming that the turbulent field is homogeneous and weak enough for triple correlations to be negligible. For initial conditions, it is postulated that the turbulence is initially isotropic, and that the turbulent magnetic field fluctuations are initially zero. The interaction of the mean magnetic field with the turbulent velocity field then causes magnetic field fluctuations to arise at later times. In general, the turbulent energy in the mechanical and magnetic modes tends toward equipartition for large values of time or of mean magnetic field. However, when the kinematic viscosity is much less than the electrical resistivity (or magnetic diffusivity), as for liquid metals, equipartition is not approached before the turbulence is damped out by the mean magnetic field.

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INTRODUCTION

TURBULENCE in conducting fluids in the presence of magnetic fields has received considerable attention in recent years.[1-5] Most of the interest in this type of turbulence stems from its importance in certain astrophysical and geophysical applications. Phenomena such as sunspots, cosmic rays, and the geomagnetic field appear to be related, respectively, to turbulence in the Sun, interstellar space, and the Earth's core. Some of the astro- and geophysical aspects of magneto-fluid dynamic turbulence are discussed in the book by Cowling.[5]

In the present study we try to gain some understanding of magneto-fluid dynamic turbulence by considering an idealized model for which a solution can be obtained. A uniform magnetic field is imposed on a field of homogeneous turbulence in a conducting fluid. Such a field of turbulence will decay with time, so that it is necessary to produce it initially by some means, for instance, by passing the fluid through a grid. Although turbulent fluctuations in the magnetic field are initially absent in such a system, it will be seen that they can arise at later times because of the interaction of the turbulent velocity field with the imposed mean magnetic field. A two point analysis of the turbulent fields will be carried out. In order to make the problem determinate, it is assumed that the turbulence is weak enough for triple correlations to be negligible. In this way deductive information on turbulence is obtained from the basic equations.[6-8] In principle the analysis could be extended to stronger turbulence by considering more points in the fluid as in reference 9.

One of the important problems in magneto-fluid dynamic turbulence is the ultimate partition of turbulent energy between the mechanical and magnetic modes. Some authors have argued that there should be an approximate equipartition of energy between the two modes, whereas others have indicated that the turbulent magnetic and vorticity fields should be similar. A summary of the arguments for and against each point of view is given in reference 5. It is hoped that the present study will give some insight into this problem. It will be seen that, at least for the model considered here, the turbulent energy tends ultimately to be distributed equally between the mechanical and magnetic modes.

BASIC EQUATIONS

It is assumed that the dynamics of the conducting fluid are described by the following equations: Navier–Stokes equation:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \frac{1}{\rho}\mathbf{j}\times\tilde{\mathbf{b}}, \tag{1}$$

Continuity equation:

$$\nabla\cdot\mathbf{u} = 0, \tag{2}$$

Maxwell's equations:

$$\mathbf{j} = \mu_0^{-1}\nabla\times\tilde{\mathbf{b}}, \tag{3}$$

$$\nabla\cdot\tilde{\mathbf{b}} = 0, \tag{4}$$

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$$\nabla \times \mathbf{E} = -\partial \tilde{\mathbf{b}}/\partial t, \tag{5}$$

Ohm's law:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \tilde{\mathbf{b}} - e_e^{-1} n_e^{-1} \mathbf{j} \times \tilde{\mathbf{b}}), \tag{6}$$

where D signifies a substantial derivative, $\mathbf{u}$ the instantaneous velocity, t the time, ρ the density, p the instantaneous pressure, $\mathbf{j}$ the instantaneous electrical current, $\tilde{\mathbf{b}}$ the instantaneous magnetic field, μ_0 the permeability of free space, $\mathbf{E}$ the instantaneous electrical field, σ the electrical conductivity, e_e the charge on an electron, and n_e the number density of electrons. mks units are used throughout. The fluid properties are assumed constant in Eqs. (1) and (6), and the usual magneto-fluid dynamic approximations are made in Maxwell's equations. The last term in Eq. (6) is, of course, the Hall current; that current arises because of the force $\mathbf{V} \times \tilde{\mathbf{b}}$ which acts on the electron gas as it moves through the fluid with the relative velocity $\mathbf{V}$. The velocity $\mathbf{V}$ is related to $\mathbf{j}$ by the equation $\mathbf{j} = -n_e e_e \mathbf{V}$.

Taking the curl of Eq. (6), and using (3), (4), and (5) results in

$$\frac{1}{\mu_0} \nabla \times (\nabla \times \tilde{\mathbf{b}}) = -\sigma \frac{\partial \tilde{\mathbf{b}}}{\partial t} + \sigma \nabla \times (\mathbf{u} \times \tilde{\mathbf{b}}) - (e_e n_e \mu_0)^{-1} \nabla \times [(\nabla \times \tilde{\mathbf{b}}) \times \tilde{\mathbf{b}}]. \tag{7}$$

With the use of the vector identities

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla \cdot \nabla \mathbf{u},$$

$$(\nabla \times \mathbf{u}) \times \mathbf{u} = (\mathbf{u} \cdot \nabla)\mathbf{u} - \tfrac{1}{2} \nabla u^2,$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u}),$$

$$\nabla \times (\nabla \mathbf{u}) = 0,$$

and Eqs. (2) and (4), Eq. (7) becomes

$$-\frac{1}{\sigma \mu_0} \nabla \cdot \nabla \tilde{\mathbf{b}} = -\frac{\partial \tilde{\mathbf{b}}}{\partial t} + \tilde{\mathbf{b}} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \tilde{\mathbf{b}} - (e_e n_e \mu_0 \sigma)^{-1} \nabla \times [(\tilde{\mathbf{b}} \cdot \nabla)\tilde{\mathbf{b}}]. \tag{8}$$

In component form this equation can be written as

$$\frac{\partial \tilde{b}_i}{\partial t} = \frac{\partial}{\partial x_k}(u_i \tilde{b}_k - \tilde{b}_i u_k) + \frac{1}{\sigma \mu_0} \frac{\partial^2 \tilde{b}_i}{\partial x_k \, \partial x_k} - (e_e n_e \mu_0 \sigma)^{-1} \epsilon_{ijk} \frac{\partial^2 (\tilde{b}_l \tilde{b}_k)}{\partial x_j \, \partial x_l}, \tag{9}$$

where ϵ_{ijk} has the value 0 when i, j, and k are not all different. When the subscripts are all different, ϵ_{ijk} has the value $+1$ when they are in cyclic order, and -1 when they are in acyclic order. The subscripts in Eq. (9) can take on the values 1, 2, or 3, and a repeated subscript in a term indicates a summation of terms, with the subscript successively taking on the values 1, 2, and 3.

The term $\mathbf{j} \times \tilde{\mathbf{b}}$ in Eq. (1) can be written as

$$\mathbf{j} \times \tilde{\mathbf{b}} = \frac{1}{\mu_0} (\nabla \times \tilde{\mathbf{b}}) \times \tilde{\mathbf{b}} = \frac{1}{\mu_0} (\tilde{\mathbf{b}} \cdot \nabla \tilde{\mathbf{b}} - \tfrac{1}{2} \nabla \tilde{b}^2).$$

Equation (1) then becomes, when written in component form,

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_k)}{\partial x_k} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \, \partial x_k} + \frac{1}{\mu_0 \rho} \frac{\partial (\tilde{b}_i \tilde{b}_k)}{\partial x_k} - \frac{1}{2\mu_0 \rho} \frac{\partial (\tilde{b}_k \tilde{b}_k)}{\partial x_i}. \tag{10}$$

Inasmuch as we will be considering a steady mean magnetic field, as well as a fluctuating field, we write the instantaneous field as

$$\tilde{b}_i = b_i + B_i \tag{11}$$

where b_i is the fluctuating component and B_i is the steady mean component of the magnetic field. Then Eq. (10) becomes

$$\frac{\partial u_i}{\partial t} = -\frac{\partial (u_i u_k)}{\partial x_k} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \, \partial x_k} + \frac{1}{\mu_0 \rho} \frac{\partial}{\partial x_k} (b_i b_k + b_i B_k + b_k B_i + B_i B_k) - \frac{1}{2\mu_0 \rho} \frac{\partial}{\partial x_i} (b_k b_k + 2 b_k B_k + B_k B_k). \tag{12}$$

The average value of Eq. (12) is

$$0 = -\frac{\partial}{\partial x_k} \overline{u_i u_k} + \frac{1}{\mu_0 \rho} \frac{\partial}{\partial x_k} (\overline{b_i b_k} + B_i B_k) - \frac{1}{2\mu_0} \frac{\partial}{\partial x_i} (\overline{b_k b_k} + B_k B_k), \tag{13}$$

where the overbars indicate average values. Subtracting Eq. (13) from Eq. (12),

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_k} (u_i u_k - \overline{u_i u_k}) - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \, \partial x_k} + \frac{1}{\mu_0 \rho} \frac{\partial}{\partial x_k} (b_i b_k - \overline{b_i b_k} + b_i B_k + b_k B_i) - \frac{1}{2\mu_0 \rho} \frac{\partial}{\partial x_i} (b_k b_k - \overline{b_k b_k} + 2 b_k B_k). \tag{14}$$

Similarly, Eq. (9) becomes

$$\frac{\partial b_i}{\partial t} = \frac{\partial}{\partial x_k} (u_i b_k - \overline{u_i b_k} + u_i B_k - b_i u_k + \overline{b_i u_k} - B_i u_k) + \frac{1}{\sigma \mu_0} \frac{\partial^2 b_i}{\partial x_k \, \partial x_k} - (e_e n_e \mu_0 \sigma)^{-1} \epsilon_{ijk} \frac{\partial^2}{\partial x_j \, \partial x_l} \cdot (b_l b_k - \overline{b_l b_k} + b_l B_k + b_k B_l). \tag{15}$$

Equations (14) and (15) are the equations for the velocity and magnetic field at a point P in the fluid. In order to construct two point correlation equations, those equations are also written at another point, say P':

$$\frac{\partial u'_i}{\partial t} = -\frac{\partial}{\partial x'_k}(u'_i u'_k - \overline{u'_i u'_k}) - \frac{1}{\rho}\frac{\partial p'}{\partial x'_i} + \nu \frac{\partial^2 u'_i}{\partial x'_k \partial x'_k} + \frac{1}{\mu_0 \rho}\frac{\partial}{\partial x'_k}(b'_i b'_k - \overline{b'_i b'_k} + b'_i B'_k + b'_k B'_i) - \frac{1}{2\mu_0\rho}\frac{\partial}{\partial x'_i}(b'_k b'_k - \overline{b'_k b'_k} + 2 b'_k B'_k), \tag{16}$$

$$\frac{\partial b'_i}{\partial t} = \frac{\partial}{\partial x'_k}(u'_i b'_k - \overline{u'_i b'_k} + u'_i B'_k - b'_i u'_k + \overline{b'_i u'_k} - B'_i u'_k) + \frac{1}{\sigma\mu_0}\frac{\partial^2 b'_i}{\partial x'_k \partial x'_k} - (e_c n_e \mu_0 \sigma)^{-1}\epsilon_{jmk}\frac{\partial^2}{\partial x'_m \partial x'_l} \cdot (b'_l b'_k - \overline{b'_l b'_k} + b'_l B'_k + b'_k B'_l). \tag{17}$$

Two-Point Correlation Equations

If we multiply Eq. (14) by u'_j and Eq. (16) by u_i, and add the two equations, and take average values, we obtain

$$\frac{\partial}{\partial t}\overline{u_i u'_j} = -\frac{\partial}{\partial x_k}\overline{u_i u_k u'_j} - \frac{\partial}{\partial x'_k}\overline{u_i u'_j u'_k} - \frac{1}{\rho}\left(\frac{\partial \overline{p u'_j}}{\partial x_i} + \frac{\partial \overline{u_i p'}}{\partial x'_j}\right) + \nu\left(\frac{\partial^2 \overline{u_i u'_j}}{\partial x_k \partial x_k} + \frac{\partial^2 \overline{u_i u'_j}}{\partial x'_k \partial x'_k}\right) + \frac{1}{\mu_0\rho}\frac{\partial}{\partial x_k}(\overline{b_i b_k u'_j} + \overline{b_i u'_j} B_k + \overline{b_k u'_j} B_i) + \frac{1}{\mu_0\rho}\frac{\partial}{\partial x'_k}(\overline{u_i b'_j b'_k} + \overline{u_i b'_j} B'_k + \overline{u_i b'_k} B'_j) - \frac{1}{2\mu_0\rho}\frac{\partial}{\partial x_i}(\overline{b_k b_k u'_j} + 2\overline{b_k u'_j} B_k) - \frac{1}{2\mu_0\rho}\frac{\partial}{\partial x'_j}(\overline{u_i b'_k b'_k} + 2\overline{u_i b'_k} B'_k). \tag{18}$$

In obtaining this equation, use was made of the fact that quantities at one point are independent of the position of the other point. Introducing the variable $r_i \equiv x'_i - x_i$ gives, for homogeneous turbulence and a uniform mean magnetic field,

$$\frac{\partial \overline{u_i u'_j}}{\partial t} = \frac{\partial}{\partial r_k}(\overline{u_i u_k u'_j} - \overline{u_i u'_j u'_k}) + \frac{1}{\mu_0\rho}\frac{\partial}{\partial r_k}(\overline{u_i b'_j b'_k} - \overline{b_i b_k u'_j}) + \frac{1}{\mu_0\rho}B_k\frac{\partial}{\partial r_k}(\overline{u_i b'_j} - \overline{b_i u'_j}) + \frac{1}{\rho}\frac{\partial}{\partial r_i}\left[\overline{p u'_j} + \frac{1}{2\mu_0}(\overline{b_k b_k u'_j} + 2\overline{b_k u'_j} B_k)\right] - \frac{1}{\rho}\frac{\partial}{\partial r_j}\left[\overline{u_i p'} + \frac{1}{2\mu_0}(\overline{u_i b'_k b'_k} + 2\overline{u_i b'_k} B_k)\right] + 2\nu\frac{\partial^2 \overline{u_i u'_j}}{\partial r_k \partial r_k}. \tag{19}$$

An equation for the correlation between the velocity and the magnetic field is obtained by multiplying Eq. (14) by b'_j, Eq. (17) by u_i, and performing operations similar to those used in obtaining Eq. (19). This gives

$$\frac{\partial \overline{u_i b'_j}}{\partial t} = \frac{\partial}{\partial r_k}\left(\overline{u_i u_k b'_j} - \frac{1}{\mu_0\rho}\overline{b_i b_k b'_j} + \overline{u_i u'_j b'_k} - \overline{u_i u'_k b'_j}\right) + B_k\frac{\partial}{\partial r_k}\left(\overline{u_i u'_j} - \frac{1}{\mu_0\rho}\overline{b_i b'_j}\right) + \frac{\partial}{\partial r_i}\left(\frac{1}{\rho}\overline{p b'_j} + \frac{1}{2\mu_0\rho}\overline{b_k b_k b'_j} + \frac{1}{\mu_0\rho}\overline{b_k b'_j} B_k\right) + \left(\nu + \frac{1}{\sigma\mu_0}\right)\frac{\partial^2 \overline{u_i b'_j}}{\partial r_k \partial r_k} - (e_c n_e \mu_0 \sigma)^{-1}\epsilon_{jmk}\frac{\partial^2}{\partial r_m \partial r_l}(\overline{u_i b'_l b'_k} + \overline{u_i b'_l} B_k + \overline{u_i b'_k} B_l). \tag{20}$$

Similarly, from Eqs. (15) and (16),

$$\frac{\partial \overline{b_i u'_j}}{\partial t} = -\frac{\partial}{\partial r_k}\left(\overline{u_i b_k u'_j} - \overline{b_i u_k u'_j} + \overline{b_i u'_j u'_k} - \frac{1}{\mu_0\rho}\overline{b_i b'_j b'_k}\right) + B_k\frac{\partial}{\partial r_k}\frac{1}{\mu_0\rho}(\overline{b_i b'_j} - \overline{u_i u'_j}) - \frac{\partial}{\partial r_j}\left(\frac{\overline{b_i p'}}{\rho} + \frac{1}{2\mu_0\rho}\overline{b_i b'_k b'_k} + \frac{1}{\mu_0\rho}\overline{b_i b'_k} B_k\right) + \left(\nu + \frac{1}{\sigma\mu_0}\right)\frac{\partial^2 \overline{b_i u'_j}}{\partial r_k \partial r_k} - (e_c n_e \mu_0 \sigma)^{-1}\epsilon_{imk}\frac{\partial^2}{\partial r_m \partial r_l} \cdot (\overline{b_l b_k u'_j} + \overline{b_l u'_j} B_k + \overline{b_k u'_j} B_l). \tag{21}$$

The equation for the two point magnetic field correlation is obtained from Eqs. (15) and (17) as

$$\frac{\partial \overline{b_i b'_j}}{\partial t} = \frac{\partial}{\partial r_k}(\overline{b_i u_k b'_j} - \overline{b_i u'_k b'_j}) + \frac{\partial}{\partial r_k}(\overline{b_i u'_j b'_k} - \overline{u_i b_k b'_j}) + B_k\frac{\partial}{\partial r_k}(\overline{b_i u'_j} - \overline{u_i b'_j}) + \frac{2}{\sigma\mu_0}\frac{\partial^2 \overline{b_i b'_j}}{\partial r_k \partial r_k} - (e_c n_e \mu_0 \sigma)^{-1}\left[\epsilon_{imk}\frac{\partial^2}{\partial r_m \partial r_l}(\overline{b_l b_k b'_j} + \overline{b_l b'_j} B_k + \overline{b_k b'_j} B_l) + \epsilon_{jmk}\frac{\partial^2}{\partial r_m \partial r_l}(\overline{b_i b'_l b'_k} + \overline{b_i b'_l} B_k + \overline{b_i b'_k} B_l)\right]. \tag{22}$$

By an argument similar to that given in reference 10, it can be shown that the term $(\partial/\partial r_k)(\overline{b_i u_k b_j'} - \overline{b_i u_k' b_j'})$ in the last equation is zero for $r = 0$. Thus, that term can be interpreted as the Fourier transform of a term which transfers magnetic energy between eddies of various sizes.

The pressure and magnetic pressure terms in Eqs. (19) to (21), that is, terms containing $\partial/\partial r_i$ or $\partial/\partial r_j$, can be interpreted as transferring energy (or correlation) between the directional components. The argument is similar to that given in reference 10. Other terms in the preceding correlation equations, or their Fourier transforms will be interpreted later in the paper.

To obtain equations for the pressure and magnetic-pressure terms, take the divergence of Eqs. (14) and (16) and apply continuity. Then Eq. (14), for instance, becomes

$$0 = -\frac{\partial^2(u_i u_k)}{\partial x_k\,\partial x_i} + \frac{\partial^2 \overline{u_i u_k}}{\partial x_k\,\partial x_i} - \frac{1}{\rho}\frac{\partial^2 p}{\partial x_i\,\partial x_i} + \frac{1}{\mu_0\rho}\left[\frac{\partial^2(b_i b_k)}{\partial x_k\,\partial x_i} - \frac{\partial^2 \overline{b_i b_k}}{\partial x_k\,\partial x_i} + \frac{\partial^2(b_i B_k)}{\partial x_k\,\partial x_i} + \frac{\partial^2(b_k B_i)}{\partial x_k\,\partial x_i}\right] - \frac{1}{2\mu_0\rho}\frac{\partial^2}{\partial x_i\,\partial x_i}(b_k b_k - \overline{b_k b_k} + 2 b_k B_k). \tag{23}$$

Multiplying this equation by u_j', averaging, and introducing the variable $\mathbf{r}$ results in

$$\frac{1}{\rho}\frac{\partial^2}{\partial r_i\,\partial r_i}\left[\overline{p u_j'} + \frac{1}{2\mu_0}(\overline{b_k b_k u_j'} + 2\overline{b_k u_j'} B_k)\right] = -\frac{\partial^2 \overline{u_i u_k u_j'}}{\partial r_i\,\partial r_k} + \frac{1}{\mu_0\rho}\frac{\partial^2 \overline{b_i b_k u_j'}}{\partial r_k\,\partial r_i}. \tag{24}$$

By taking the Fourier transform of this equation, it is easy to see that the quantity in brackets will be zero if the triple correlations are neglected. [See for instance Eq. (14), reference 9.] Similarly all of the other pressure and magnetic-pressure terms in Eqs. (19) to (21) will be zero if the triple correlations are neglected. Thus neglecting triple correlations and assuming that the mean magnetic field is in the direction B_3, the set of Eqs. (19) to (22) becomes

$$\frac{\partial \overline{u_i u_j'}}{\partial t} = \frac{1}{\mu_0\rho} B_3 \frac{\partial}{\partial r_3}(\overline{u_i b_j'} - \overline{b_i u_j'}) + 2\nu \frac{\partial^2 \overline{u_i u_j'}}{\partial r_k\,\partial r_k}, \tag{25}$$

$$\frac{\partial \overline{u_i b_j'}}{\partial t} = B_3 \frac{\partial}{\partial r_3}\left(\overline{u_i u_j'} - \frac{1}{\mu_0\rho}\overline{b_i b_j'}\right) + \left(\nu + \frac{1}{\sigma\mu_0}\right)\frac{\partial^2 \overline{u_i b_j'}}{\partial r_k\,\partial r_k} - (e_c n_e \mu_0 \sigma)^{-1} B_3\left(\epsilon_{jm3}\frac{\partial^2 \overline{u_i b_l'}}{\partial r_m\,\partial r_l} + \epsilon_{jmk}\frac{\partial^2 \overline{u_i b_k'}}{\partial r_m\,\partial r_3}\right), \tag{26}$$

$$\frac{\partial \overline{b_i u_j'}}{\partial t} = B_3 \frac{\partial}{\partial r_3}\left(\frac{1}{\mu_0\rho}\overline{b_i b_j'} - \overline{u_i u_j'}\right) + \left(\nu + \frac{1}{\sigma\eta_0}\right)\frac{\partial^2 \overline{b_i u_j'}}{\partial r_k\,\partial r_k} - (e_c n_e \mu_0 \sigma)^{-1} B_3\left(\epsilon_{im3}\frac{\partial^2 \overline{b_l u_j'}}{\partial r_m\,\partial r_l} + \epsilon_{imk}\frac{\partial^2 \overline{b_k u_j'}}{\partial r_m\,\partial r_3}\right), \tag{27}$$

$$\frac{\partial \overline{b_i b_j'}}{\partial t} = B_3 \frac{\partial}{\partial r_3}(\overline{b_i u_j'} - \overline{u_i b_j'}) + \frac{2}{\sigma\mu_0}\frac{\partial^2 \overline{b_i b_j'}}{\partial r_k\,\partial r_k} - (e_c n_e \mu_0 \sigma)^{-1} B_3\left(\epsilon_{im3}\frac{\partial^2 \overline{b_l b_j'}}{\partial r_m\,\partial r_l} + \epsilon_{imk}\frac{\partial^2 \overline{b_k b_j'}}{\partial r_m\,\partial r_3} + \epsilon_{jm3}\frac{\partial^2 \overline{b_i b_l'}}{\partial r_m\,\partial r_l} + \epsilon_{jmk}\frac{\partial^2 \overline{b_i b_k'}}{\partial r_m\,\partial r_3}\right). \tag{28}$$

Spectral Equations

In order to convert Eqs. (25) to (28) to spectral form, define the following three-dimensional Fourier transforms:

$$\overline{u_i u_j'} = \int_{-\infty}^{\infty} \varphi_{ij} e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}\, d\boldsymbol{\kappa}, \tag{29}$$

$$\overline{b_i b_j'} = \int_{-\infty}^{\infty} \beta_{ij} e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}\, d\boldsymbol{\kappa}, \tag{30}$$

$$\overline{b_i u_j'} = \int_{-\infty}^{\infty} \beta_{ij}' e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}\, d\boldsymbol{\kappa}, \tag{31}$$

$$\overline{u_i b_j'} = \int_{-\infty}^{\infty} \beta_{ij}'' e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}\, d\boldsymbol{\kappa}. \tag{32}$$

By using these relations, the Fourier transforms of Eqs. (25) to (28) are obtained as

$$\frac{\partial \varphi_{ij}}{\partial t} = \frac{1}{\mu_0\rho} B_3 i\kappa_3(\beta_{ij}'' - \beta_{ij}') - 2\nu\kappa^2\varphi_{ij}, \tag{33}$$

$$\frac{\partial \beta_{ij}''}{\partial t} = B_3 i\kappa_3\left(\varphi_{ij} - \frac{1}{\mu_0\rho}\beta_{ij}\right) - \left(\nu + \frac{1}{\sigma\mu_0}\right)\kappa^2\beta_{ij}'' + (e_c n_e \mu_0\sigma)^{-1} B_3(\epsilon_{jm3}\kappa_m\kappa_l\beta_{il}'' + \epsilon_{jmk}\kappa_m\kappa_3\beta_{ik}''), \tag{34}$$

$$\frac{\partial \beta_{ij}'}{\partial t} = B_3 i\kappa_3\left(\frac{1}{\mu_0\rho}\beta_{ij} - \varphi_{ij}\right) - \left(\nu + \frac{1}{\sigma\mu_0}\right)\kappa^2\beta_{ij}' + (e_c n_e \mu_0\sigma)^{-1} B_3(\epsilon_{im3}\kappa_m\kappa_l\beta_{lj}' + \epsilon_{imk}\kappa_m\kappa_3\beta_{kj}'), \tag{35}$$

$$\frac{\partial \beta_{ij}}{\partial t} = B_3 i\kappa_3(\beta_{ij}' - \beta_{ij}'') - \frac{2\kappa^2}{\sigma\mu_0}\beta_{ij} + (e_c n_e \mu_0\sigma)^{-1} B_3(\epsilon_{im3}\kappa_m\kappa_l\beta_{lj} + \epsilon_{imk}\kappa_m\kappa_3\beta_{kj} + \epsilon_{jm3}\kappa_m\kappa_l\beta_{il} + \epsilon_{jmk}\kappa_m\kappa_3\beta_{ik}). \tag{36}$$

As they stand, most of the 36 equations represented by (33) to (36) are interrelated. The solution could be carried out numerically if it appears to be important to do so. However, if the Hall current terms are neglected (large n_e or σ), a considerable simplification is obtained, inasmuch as it is then only necessary to solve three equations simultaneously.

[10] G. K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge University Press, New York, 1953), p. 87.

The Hall currents are negligible for a liquid metal, although they may not be for a rarified plasma. It appears that the results will still be useful for giving an insight into some of the physical processes occurring in this type of turbulence.

If we neglect Hall currents and compare Eqs. (34) and (35), we find that the relation $\beta''_{ij} = -\beta'_{ij}$ holds for all times if it is true at an initial time. Here it will be assumed that the magnetic field fluctuations are initially zero, so that the above relation will hold. Thus, the set of Eqs. (33) to (36) becomes

$$\frac{\partial \varphi_{ij}}{\partial t} = -\left(\frac{i2B_3\kappa_3\beta'_{ij}}{\mu_0\rho}\right) - 2\nu\kappa^2\varphi_{ij}, \tag{37}$$

$$\frac{\partial}{\partial t}\left(\frac{i2B_3\kappa_3\beta'_{ij}}{\mu_0\rho}\right) = \frac{2B_3^2\kappa_3^2}{\mu_0\rho}\left[\varphi_{ij} - \left(\frac{\beta_{ij}}{\mu_0\rho}\right)\right] - \left(\nu + \frac{1}{\sigma\mu_0}\right)\kappa^2\left(\frac{i2B_3\kappa_3\beta'_{ij}}{\mu_0\rho}\right), \tag{38}$$

$$\frac{\partial}{\partial t}\left(\frac{\beta_{ij}}{\mu_0\rho}\right) = \left(\frac{i2B_3\kappa_3\beta'_{ij}}{\mu_0\rho}\right) - \frac{2\nu\kappa^2}{\sigma\mu_0\nu}\left(\frac{\beta_{ij}}{\mu_0\rho}\right). \tag{39}$$

In these equations φ_{ij} is a spectrum tensor for turbulent velocity energy and $\beta_{ij}/(\mu_0\rho)$ is a corresponding tensor for turbulent magnetic energy. The term $i2B_3\kappa_3\beta'_{ij}/(\mu_0\rho)$ occurs in both Eqs. (37) and (39), but with opposite signs. Thus it can be interpreted as an interchange term which transfers turbulent energy between the mechanical and magnetic modes.

Solution of Spectral Equations

A general solution of Eqs. (37) to (39) is

$$\varphi_{ij} = \exp\left[-\left(\nu + \frac{1}{\sigma\mu_0}\right)\kappa^2(t - t_0)\right] \cdot \{(C_1)_{ij} + (C_2)_{ij}\exp[s(t - t_0)] + (C_3)_{ij}\exp[-s(t - t_0)]\}, \tag{40}$$

$$\frac{i2B_3\kappa_3\beta'_{ij}}{\mu_0\rho} = \exp\left[-\left(\frac{1}{\sigma\mu_0} + \nu\right)\kappa^2(t - t_0)\right] \cdot\left\{(C_1)_{ij}\left(\frac{1}{\sigma\mu_0} - \nu\right)\kappa^2 + (C_2)_{ij}\left[\left(\frac{1}{\sigma\mu_0} - \nu\right)\kappa^2 - s\right]\exp[s(t - t_0)] + (C_3)_{ij}\left[\left(\frac{1}{\sigma\mu_0} - \nu\right)\kappa^2 + s\right]\exp[-s(t - t_0)]\right\}, \tag{41}$$

$$\frac{\beta_{ij}}{\mu_0\rho} = \frac{1}{\mu_0\rho}\exp\left[-\left(\frac{1}{\sigma\mu_0} + \nu\right)\kappa^2(t - t_0)\right]\left\{\frac{2(C_1)_{ij}B_3^2\kappa_3^2}{\mu_0\rho} + (C_2)_{ij}\left[\left(\frac{1}{\sigma\mu_0} - \nu\right)^2\kappa^4 - \left(\frac{1}{\sigma\mu_0} - \nu\right)\kappa^2 s - \frac{2B_3^2\kappa_3^2}{\mu_0\rho}\right] \cdot\exp[s(t - t_0)] + (C_3)_{ij}\left[\left(\frac{1}{\sigma\mu_0} - \nu\right)^2\kappa^4 + \left(\frac{1}{\sigma\mu_0} - \nu\right)\kappa^2 s - \frac{2B_3^2\kappa_3^2}{\mu_0\rho}\right] \cdot\exp[-s(t - t_0)]\right\} \Big/ \left(\frac{2B_3^2\kappa_3^2}{\mu_0^2\rho^2}\right), \tag{42}$$

where

$$s \equiv \left[\left(\frac{1}{\sigma\mu_0} - \nu\right)^2\kappa^4 - \frac{4B_3^2\kappa_3^2}{\mu_0\rho}\right]^{\frac{1}{2}}, \tag{43}$$

and $(C_1)_{ij}$, $(C_2)_{ij}$, and $(C_3)_{ij}$ are constants of integration.

In order to evaluate the constants of integration, we use the conditions that the turbulence is initially isotropic and that β_{ij} and $i\beta'_{ij}$ are initially zero. The last two conditions correspond to the assumption that the magnetic field fluctuations are zero at $t = t_0$. The interaction of the mean magnetic field and the velocity fluctuations will then cause magnetic field fluctuations to arise at later times, and, in addition, will cause the turbulence to become anisotropic. The assumption that the turbulence is initially isotropic means that, for weak turbulence,

$$(\varphi_{ij})_0 = (J_0/12\pi^2)(\kappa^2\delta_{ij} - \kappa_i\kappa_j) \tag{44}$$

as given by Eq. (43) in reference 6. The quantity J_0 is a constant that depends on initial conditions and δ_{ij} is the Kronecker delta. For the foregoing initial conditions, the constants of integration are

$$(C_1)_{ij} = -\frac{J_0\kappa^2}{6\pi^2 s^2}\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right)\frac{B_3^2\kappa_3^2}{\mu_0\rho}, \tag{45}$$

$$(C_2)_{ij} = \frac{J_0\kappa^2}{24\pi^2 s^3}\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right)\left[\left(\frac{1}{\sigma\mu_0} - \nu\right)^3\kappa^6 + \left(\frac{1}{\sigma\mu_0} - \nu\right)^2\kappa^4 s - 4\left(\frac{1}{\sigma\mu_0} - \nu\right)\frac{\kappa^2 B_3^2\kappa_3^2}{\mu_0\rho} - \frac{2B_3^2\kappa_3^2 s}{\mu_0\rho}\right], \tag{46}$$

and

$$(C_3)_{ij} = \frac{-J_0\kappa^2}{24\pi^2 s^3}\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right)\left[\left(\frac{1}{\sigma\mu_0} - \nu\right)^3\kappa^6 - \left(\frac{1}{\sigma\mu_0} - \nu\right)^2\kappa^4 s - 4\left(\frac{1}{\sigma\mu_0} - \nu\right)\frac{\kappa^2 B_3^2\kappa_3^2}{\mu_0\rho} + \frac{2B_3^2\kappa_3^2 s}{\mu_0\rho}\right]. \tag{47}$$

When s, as given by Eq. (43), becomes imaginary the following solution can be used:

$$\varphi_{ij} = \{(C'_1)_{ij} + (C'_2)_{ij}\cos[s'(t - t_0)] + (C'_3)_{ij} \cdot\sin[s'(t - t_0)]\}\exp\left[-\left(\frac{1}{\sigma\mu_0} + \nu\right)\kappa^2(t - t_0)\right], \tag{48}$$

$$\frac{i2B_3\kappa_3\beta'_{ij}}{\mu_0\rho} = \exp\left[-\left(\frac{1}{\sigma\mu_0}+\nu\right)\kappa^2(t-t_0)\right] \cdot\left\{(C'_1)_{ij}\left(\frac{1}{\sigma\mu_0}-\nu\right)\kappa^2 + \left[(C'_2)_{ij}\left(\frac{1}{\sigma\mu_0}-\nu\right)\kappa^2 - (C'_3)_{ij}s'\right]\cos[s'(t-t_0)] + \left[(C'_3)_{ij}\left(\frac{1}{\sigma\mu_0}-\nu\right)\kappa^2 + (C'_2)_{ij}s'\right]\cdot\sin[s'(t-t_0)]\right\}, \tag{49}$$

$$\frac{\beta_{ij}}{\mu_0\rho} = \frac{1}{\mu_0\rho}\exp\left[-\left(\frac{1}{\sigma\mu_0}+\nu\right)\kappa^2(t-t_0)\right]\left\{\frac{2(C'_1)_{ij}B_3^2\kappa_3^2}{\mu_0\rho} + \left[(C'_2)_{ij}\left(\frac{1}{\sigma\mu_0}-\nu\right)^2\kappa^4 - (C'_3)_{ij}s'\left(\frac{1}{\sigma\mu_0}-\nu\right)\kappa^2 - \frac{2(C'_2)_{ij}B_3^2\kappa_3^2}{\mu_0\rho}\right]\cos[s'(t-t_0)] + \left[(C'_3)_{ij}\left(\frac{1}{\sigma\mu_0}-\nu\right)^2\kappa^4 + (C'_2)_{ij}s'\left(\frac{1}{\sigma\mu_0}-\nu\right)\kappa^2 - \frac{2(C'_3)_{ij}B_3^2\kappa_3^2}{\mu_0\rho}\right]\sin[s'(t-t_0)]\right\}\Big/\left(\frac{2B_3^2\kappa_3^2}{\mu_0^2\rho^2}\right), \tag{50}$$

where

$$s' = \left[\frac{4B_3^2\kappa_3^2}{\mu_0\rho} - \left(\frac{1}{\sigma\mu_0}-\nu\right)^2\kappa^4\right]^{\frac{1}{2}}, \tag{51}$$

$$(C'_1)_{ij} = \frac{J_0\kappa^2}{6\pi^2 s'^2}\left(\delta_{ij}-\frac{\kappa_i\kappa_j}{\kappa^2}\right)\frac{B_3^2\kappa_3^2}{\mu_0\rho}, \tag{52}$$

$$(C'_2)_{ij} = \frac{J_0\kappa^2}{12\pi^2 s'^2}\left(\delta_{ij}-\frac{\kappa_i\kappa_j}{\kappa^2}\right)\cdot\left[\frac{2B_3^2\kappa_3^2}{\mu_0\rho} - \left(\frac{1}{\sigma\mu_0}-\nu\right)^2\kappa^4\right], \tag{53}$$

and

$$(C'_3)_{ij} = \frac{J_0\kappa^2}{12\pi^2 s'}\left(\delta_{ij}-\frac{\kappa_i\kappa_j}{\kappa^2}\right)\left(\frac{1}{\sigma\mu_0}-\nu\right)\kappa^2. \tag{54}$$

The foregoing spectral quantities are functions of the components of $\boldsymbol{\kappa}$ as well as of its magnitude. In order to obtain quantities that are functions only of the magnitude κ we integrate over all directions in wavenumber space. Thus, as suggested by Batchelor, we define the quantity ψ_{ij} by the equation

$$\psi_{ij}(\kappa) = \int_0^A \varphi_{ij}(\boldsymbol{\kappa})\,dA(\boldsymbol{\kappa}) \tag{55}$$

where A is the area of a sphere of radius κ. Letting $r = 0$ in Eq. (29) shows that

$$\overline{u_iu_j} = \int_0^\infty \psi_{ij}\,d\kappa \tag{56}$$

Thus $\psi_{ij}\,d\kappa$ gives the contribution from wave number band $d\kappa$ to $\overline{u_iu_j}$, and a plot of ψ_{ij} against κ shows how contributions to $\overline{u_iu_j}$ are distributed among the various wave numbers or eddy sizes.

Equations (40) to (54) can be written in spherical coordinates by setting

$$\kappa_1 = \kappa\cos\varphi\sin\theta,\quad \kappa_2 = \kappa\sin\varphi\sin\theta,\quad \kappa_3 = \kappa\cos\theta. \tag{57}$$

Then Eq. (55) becomes

$$\psi_{ij}(\kappa) = \int_0^\pi\int_0^{2\pi}\varphi_{ij}(\kappa,\varphi,\theta)\kappa^2\sin\theta\,d\varphi\,d\theta. \tag{58}$$

The component φ_{33} and φ_{ii} are independent of the angle φ. Thus Eq. (58) becomes, for ψ_{ii},

$$\psi_{ii} = 4\pi\kappa^2\int_0^1\varphi_{ii}\,d(\cos\theta). \tag{59}$$

Spectrum functions corresponding respectively to β_{ii}, and to the interchange term in Eqs. (37) and (39), are

$$\Omega_{ii} = 4\pi\kappa^2\int_0^1\beta_{ii}\,d(\cos\theta), \tag{60}$$

and

$$I_{ii} = 4\pi\kappa^2\int_0^1 (i2B_3\kappa_3\beta'_{ii}/\mu_0\rho)\,d(\cos\theta). \tag{61}$$

Similar expressions are used to calculate ψ_{33} and Ω_{33}. Computed spectra of the various turbulent quantities will be considered in the next section. For making the calculations, the indicated integrations in Eqs. (59), (60), and (61) were carried out numerically.

RESULTS AND DISCUSSION

Figure 1 shows dimensionless spectra of the energy contained in the turbulent velocity field (spectra of $\frac{1}{2}\overline{u_iu_i}$). Curves are shown for values of $\sigma\mu_0\nu$ of 10^{-7}, 0.5, and 2. The low value of $\sigma\mu_0\nu$ corresponds, in order of magnitude, to liquid metals, whereas the higher values may approximate those for certain rarefied astrophysical plasmas.

When plotted with the similarity variables used in Fig. 1, the spectrum for no magnetic effects ($B_3^* = 0$) does not change with time, so that comparison of the various curves indicates how magnetic forces will alter the spectrum. In all cases the curves for $B_3^* \neq 0$ lie below those for $B_3^* = 0$, thus illustrating the stabilizing effect of the magnetic field. For the two lower values of $\sigma\mu_0\nu$, the areas under

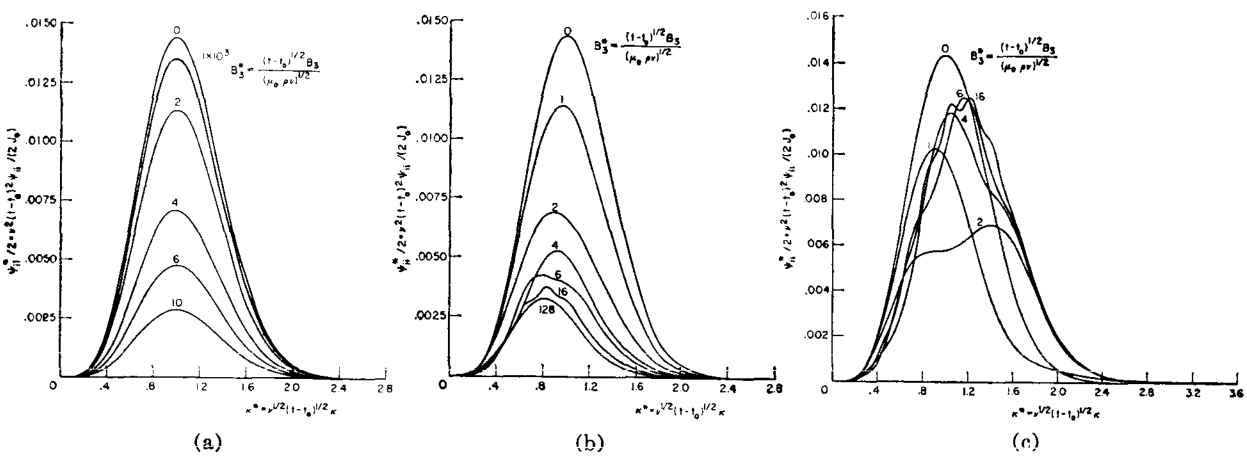

FIG. 1. Dimensionless spectra of energy in turbulent velocity field, $\frac{1}{2}\overline{u_i u_i}$. (a) $\sigma\mu_0\nu = 10^{-7}$. (b) $\sigma\mu_0\nu = 0.5$. (c) $\sigma\mu_0\nu = 2.0$.

the curves decrease as B_3^* increases. However, for $\sigma\mu_0\nu = 2$, a minimum is indicated, and further increases in B_3^* cause a relative increase in turbulent velocity energy. These effects are shown more clearly in Fig. 2 where values of $\overline{u_i u_i}/(\overline{u_i u_i})_0$ and $\overline{u_3 u_3}/(\overline{u_3 u_3})_0$ are plotted against B_3^*. The subscript 0 is for $B_3^* = 0$. The ordinates were obtained from the areas under spectrum curves such as those in Fig. 1 and similar curves for ψ_{33}. For high values of B_3^* the velocity fluctuations for $\sigma\mu_0\nu = 2$ are nearly as strong as those for $B_3^* = 0$, although the spectra differ. For $\sigma\mu_0\nu < 1$, the velocity fluctuations decrease and approach zero for large values of B_3^*, the rate of approach to zero being much greater for the lower value of $\sigma\mu_0\nu$.

This damping of the velocity fluctuations seems to be related to the darkness of sunspots, as discussed by Cowling (reference 5). The magnetic field in the sunspot apparently reduces the turbulence in that region, and thus reduces the convective heat transport to the surface. Thus the surface appears dark. Inasmuch as $\sigma\mu_0\nu$ is less than 1 for the Sun, Fig. 2 indicates that this damping of the turbulence

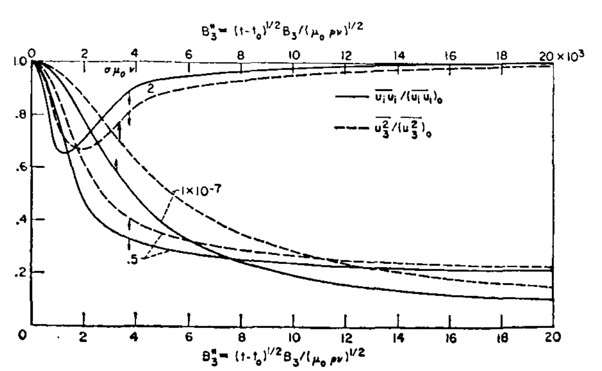

FIG. 2. Ratio of mean turbulent velocity fluctuations to those for no magnetic field.

should take place if the mean magnetic field in the sunspot is sufficiently large.

The energy in the turbulent magnetic field will be considered next. Dimensionless spectra of $\frac{1}{2}\overline{b_i b_i}$ are plotted in Fig. 3, where it is seen that the trends for small B_3^* are opposite to those for the turbulent velocity field shown in Fig. 1. As B_3^* increases, the turbulent magnetic energy, in general, increases relative to the turbulent velocity energy for no magnetic field. (The comparison is made relative to the velocity energy for $B_3^* = 0$ since the turbulent magnetic energy is zero for $B_3^* = 0$.) For larger values of B_3^* the variation is more complex. However, it appears that for all values of B_3^*, turbulent energy is being transferred between the mechanical mode and the magnetic mode by the interchange term in Eqs. (37) and (39).

The integrated interchange term as calculated by Eq. (61) is plotted in Fig. 4. That term is predominantly positive for $\sigma\mu_0\nu < 1$, and thus indicates, that turbulent energy is being transferred from the mechanical to the magnetic mode. For $\sigma\mu_0\nu = 2$ and low values of B_3^* (not shown) the interchange term is also predominantly positive. However, for $B_3^* = 16$ [Fig. 4(c)], I_{ii} is predominantly negative. That is, for $\sigma\mu_0\nu > 1$, energy is transferred from the magnetic to the mechanical mode at high values of B_3^*. Energy can apparently be transferred from the velocity to the fluctuating magnetic mode because the magnetic lines of force in the mean imposed magnetic field tend to follow the fluid motions, at least at high values of conductivity or of $\sigma\mu_0\nu$. (The turbulent velocity fluctuations tend to scramble the magnetic lines of force.[4]) If, on the other hand, the fluctuating magnetic energy happens to be very large, the tension in the lines of force tends to straighten them out and reduces the fluctuating

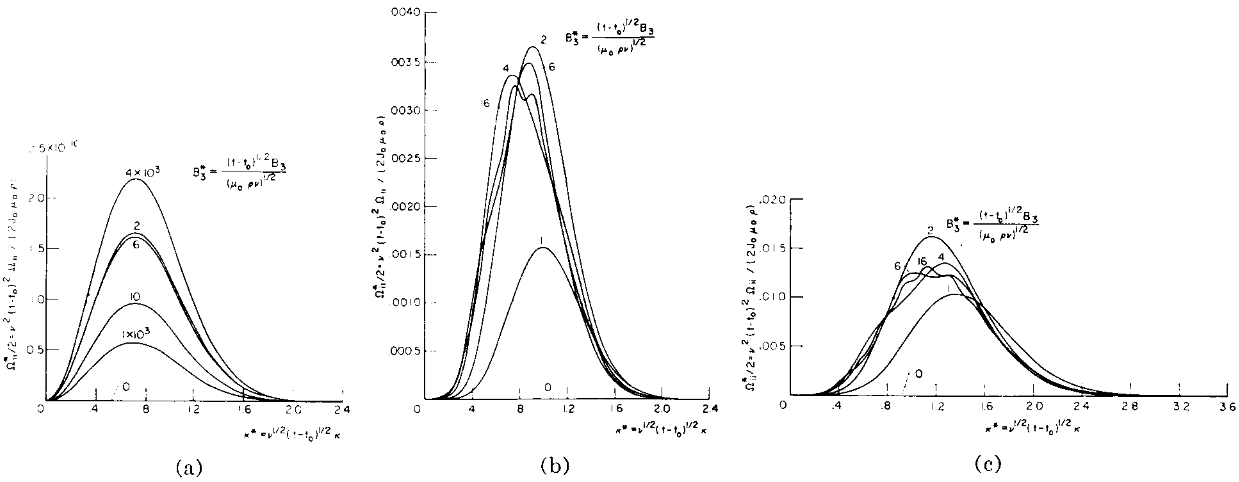

FIG. 3. Dimensionless spectra of energy in turbulent magnetic field, $\frac{1}{2}\overline{b_ib_i}$. (a) $\sigma\mu_0\nu = 10^{-7}$. (b) $\sigma\mu_0\nu = 0.5$. (c) $\sigma\mu_0\nu = 2.0$.

magnetic energy. Thus energy can be transferred in both directions.

Some of the curves in Figs. 4(b) and (c) exhibit multiple peaks and valleys, the effect being particularly pronounced for $\sigma\mu_0\nu = 2$ and $B_3^* = 16$. For that case, Fig. 4(c) indicates that the energy transfer can be in one direction at a given wave number and in the opposite direction at a slightly different wave number. A similar curve was obtained for $\sigma\mu_0\nu = 0.5$ and $B_3^* = 16$, but the curve in that case was predominantly positive, rather than negative. Because of the interaction of Eqs. (37) to (39), some of the curves in Figs. 1 and 3 also have a wavy appearance. In all cases the number of peaks increases as B_3^* increases. In the curve in Fig. 1(b) for $\sigma\mu_0\nu = 0.5$ and $B_3^* = 128$, there are actually a large number of very small peaks, but they are too small to show up in the plot. The quantity B_3^* is proportional to $(t - t_0)^{\frac{1}{2}}$, so that as time increases, for a fixed B_3, the number of peaks in the spectrum curves will increase.

A necessary condition for the development of multiple peaks seems to be that the energy in the mechanical and magnetic modes be of the same order of magnitude. Thus the effect does not occur for $\sigma\mu_0\nu = 10^{-7}$, or for other values of $\sigma\mu_0\nu$ when B_3^* is small, inasmuch as the energy in the magnetic mode is much less than that in the mechanical mode for those cases. The second term in Eq. (38) is proportional to the difference between the mechanical and magnetic energy at a particular value of $\boldsymbol{\kappa}$. It seems reasonable that the waviness observed in the spectra should develop when the two quantities are of the same order of magnitude.

Comparison of Figs. 1(a) and 3(a) indicates that negligible energy is contained in the magnetic mode for $\sigma\mu_0\nu = 10^{-7}$. This occurs in spite of the fact that the interchange terms for $\sigma\mu_0\nu = 10^{-7}$ and for

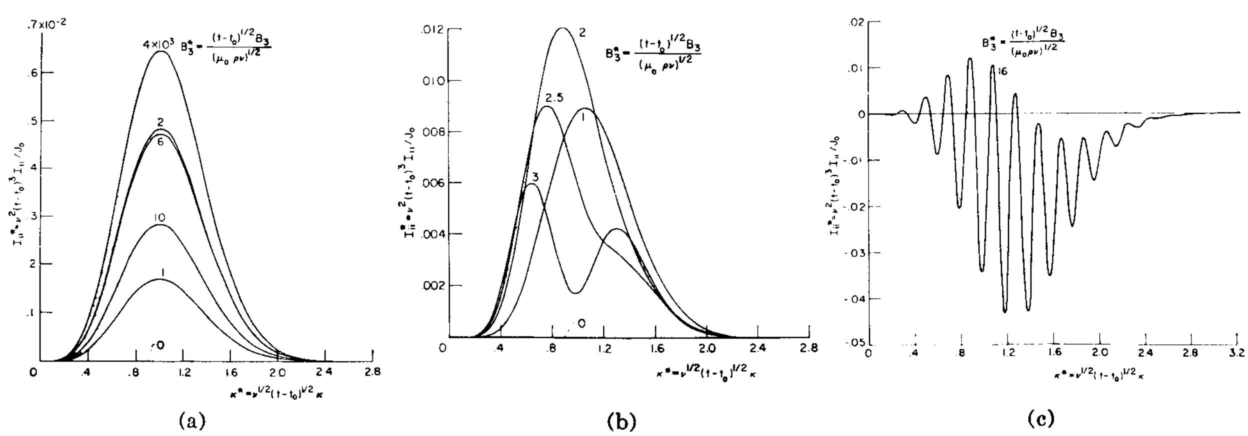

FIG. 4. Dimensionless interchange term for transfer of energy between mechanical and magnetic modes. (a) $\sigma\mu_0\nu = 10^{-7}$. (b) $\sigma\mu_0\nu = 0.5$. (c) $\sigma\mu_0\nu = 2.0$.

0.5 are of the same order of magnitude [see Figs. 4(a) and (b)]. Comparison of the interchange term and electrical dissipation terms in Eq. (39) shows that they are very nearly equal. Thus, for $\sigma\mu_0\nu = 10^{-7}$, nearly all the energy that is transferred out of the mechanical mode is dissipated immediately by electrical resistance. The energy dissipated by electrical resistance is of the same order of magnitude as that dissipated by viscous action for all three values of $\sigma\mu_0\nu$, as can be seen by comparison of Figs. 1 and 3 and the dissipation terms in Eqs. (37) and (39). The ratio of the dissipation term for magnetic energy to that for mechanical energy (integrated over all directions) is the same as the ratio of an ordinate on Fig. 3 to a corresponding ordinate on Fig. 1 divided by $\sigma\mu_0\nu$.

Partition of Turbulent Energy Between Velocity and Magnetic Fields

A point of considerable physical interest is the partition of turbulent energy between the mechanical and magnetic modes. Figure 5 shows $\overline{b_ib_i}/(\mu_0\rho\overline{u_iu_i})$ and $\overline{b_3^2}/\mu_0\rho\overline{u_3^2}$ plotted against B_3^*. For the two higher values of $\sigma\mu_0\nu$, the curves approach one for large values of time or of B_3. On the other hand, for $\sigma\mu_0\nu = 10^{-7}$, the ratios are essentially 0 for values of B_3^* up to 130×10^3. For the higher values of $\sigma\mu_0\nu$, appreciable turbulent energy remains in the fluid when equipartition of energy is approached, but for $\sigma\mu_0\nu = 10^{-7}$ the turbulent energy is damped out by the mean magnetic field without approaching equipartition (see Figs. 2 and 5). The small amount of fluctuating magnetic energy in this case can be thought of as due to the fact that the mean magnetic lines of force are not pulled about by the velocity field except for fluids of very high conductivity (high $\sigma\mu_0\nu$).

The tendency of the energy to approach equipartition can be explained with the aid of Eqs. (37) to (39). The second term in Eq. (38) will produce a positive contribution to the change in the transfer term when the mechanical energy is greater than the magnetic energy and vice versa. Reference to Eqs. (37) and (39) then shows that its effect will be to produce equality of energy in the mechanical and magnetic modes. For $\sigma\mu_0\nu = 2$, the ratio of magnetic to mechanical energy becomes greater than 1 before leveling off at unity. This evidently can occur because the dissipation rate for mechanical energy is greater than that for magnetic energy in that case.

It might be emphasized that the present results concerning the equipartition of energy are for a weak turbulence in which triple correlations are negligible. However, regardless of the effect which triple correlations may have on the partition of energy, the terms in Eqs. (37) to (39) which tend to produce equipartition will still be present if a mean magnetic field is imposed. In connection with the triple correlations, it is of interest that they will not arise because of the interaction of the magnetic and velocity fields if those correlations are initially absent. For if we construct three point equations for the triple correlations from Eqs. (14) and (15) by multiplying them through by velocity or magnetic field fluctuations at two other points, we find, for instance, that $\partial\overline{u_iu_i'u_k''}/\partial t = 0$ if the triple and quadruple correlations are initially zero. (Note that terms such as $\partial\overline{u_iu_k}/\partial x_k$ are zero for homogeneous turbulence.) This is in contrast to the case of the double correlations $\overline{u_ib_i'}$. Those correlations can arise even though initially zero, so long as $\overline{u_iu_i'}$ is not initially zero [Eq. (20)].

It has been suggested that cosmic rays are caused by the acceleration of charged particles by wandering (or turbulent) magnetic fields in interstellar space.[11] Inasmuch as $\sigma\mu_0\nu$ is probably greater than one for this case, Fig. 5 indicates that the required turbulent magnetic field should exist if a turbulent velocity field and a mean magnetic field are present. Inasmuch as $\sigma\mu_0\nu$ is probably greater than one in interstellar space, the energy in the turbulent magnetic field should be at least as great as that in the velocity field.

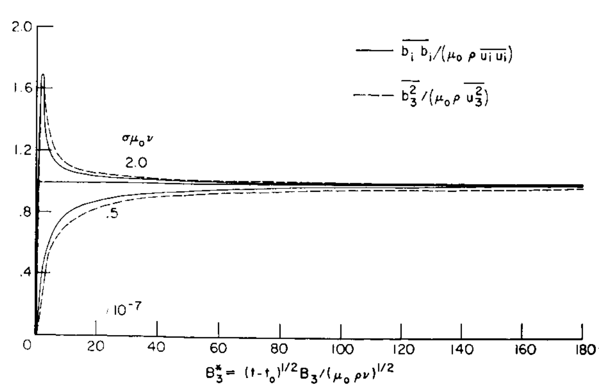

FIG. 5. Ratio of turbulent energy in magnetic mode to that in mechanical mode.

Partition of Turbulent Energy Between Directional Components

Another point of interest is the partition of the mechanical and magnetic energy between the directional components. Values of $\overline{u_3^2}/(\frac{1}{3}\overline{u_iu_i})$ and of

[11] E. Fermi, Phys. Rev. 75, 1169 (1949).

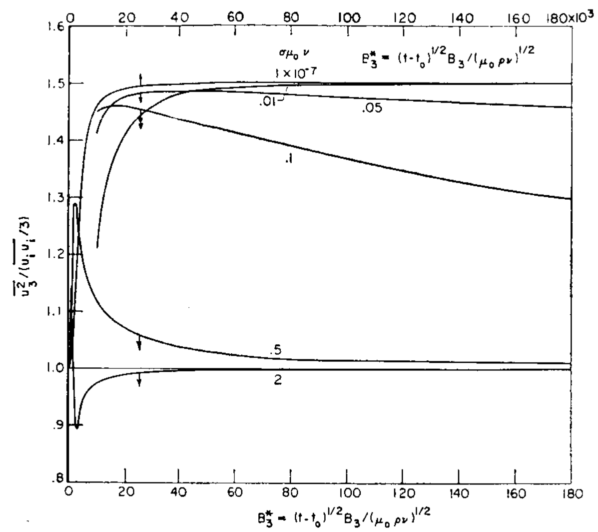

FIG. 6. Ratio of component of turbulent mechanical energy in direction of mean magnetic field to average directional component.

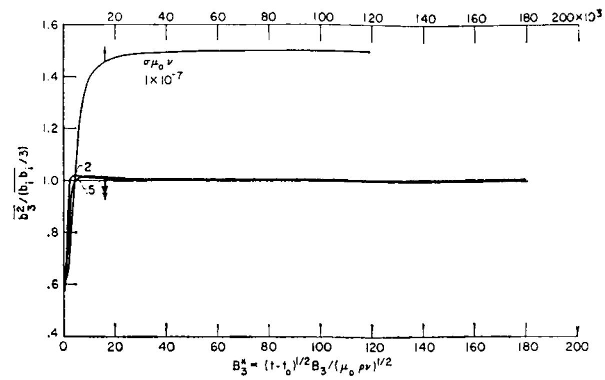

FIG. 7. Ratio of component of turbulent magnetic energy in direction of mean magnetic field to average directional component.

$\overline{b_3^2}/(\frac{1}{3}\overline{b_ib_i})$ are plotted against B_3^* in Figs. 6 and 7. A value of one on the ordinates of these curves indicates that the three directional components of the mechanical and the magnetic energy are equal, inasmuch as $\overline{u_1^2} = \overline{u_2^2}$ and $\overline{b_1^2} = \overline{b_2^2}$ from symmetry considerations. The curves indicate considerable interchange between the directional components. This may be somewhat surprising in view of the fact that the pressure force terms, which are usually associated with the transfer between directional components, are absent in the present case. The interchange might be attributed to a difference in the decay rates and in the transfer between the magnetic and mechanical modes for the three directional components.

For large values of B_3^* the curves for $\sigma\mu_0\nu = 0.5$ and 2 approach one. Those for lower values of $\sigma\mu_0\nu$ tend to approach 1.5 and then, except for $\sigma\mu_0\nu = 10^{-2}$ and 10^{-7} decrease toward one. Equal energy can evidently occur in the three directional components because each of Eqs. (37) to (39) is identical for the three components. Thus, if the effects of initial conditions as given by Eq. (44) are negligible for large values of B_3^* the three components of the mechanical and the magnetic energy could be equal, as shown in Figs. 6 and 7.

CONCLUSIONS

Turbulent energy is transferred between the mechanical and magnetic modes by an interchange term in the spectral and correlation equations, this term being proportional to the mean magnetic field. A term in the equation of change for the interchange term is proportional to the difference between the mechanical and magnetic energy and causes energy to be transferred in such a direction that it tends to produce equipartition of energy between the mechanical and magnetic modes. Multiple peaks can develop in the spectrum curves when the energy in the mechanical and magnetic modes is of the same order of magnitude. When the kinematic viscosity is much less than the electrical resistivity (or magnetic diffusivity) as for liquid metals, most of the energy transferred out of the mechanical mode is dissipated immediately by electrical resistance. Thus very little energy resides in the magnetic mode, and the interchange term, in this case, is ineffective in producing equipartition of energy. Except for very low ratios of kinematic viscosity to electrical resistivity, the turbulent energy in the velocity and magnetic fields tends to be equally divided between the directional components for large values of time or of mean magnetic field.

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